Fractals in surface growth with power-law noise

1992 J. Phys. A: Math. Gen. 25 L135

(http://iopscience.iop.org/0305-4470/25/3/009)

J. Phys. A: Math. Gen. 25 (1992) L135-L140. Printed in the UK

LETTER TO THE EDITOR

Fractals in surface growth with power-law noise

Chi-Hang Lam and Leonard M Sander
Randall Laboratory of Physics, University of Michigan, Ann Arbor, MI 48109, USA

Received 3 August 1991

Abstract. We present a microscopic description of interface growth with power-law noise distribution in the form $P(\eta)\sim 1/\eta^{1+\mu}$, which exhibits non-universal roughening. For the $\mu=d+1$ case in $d+1$ dimensions, the existence of a fractal pattern in the *bulk* of the aggregate is explained, leading trivially to the proof of the identity $\alpha+z=2$ for the roughening and the dynamical scaling exponents α and z respectively. Investigations on the distribution of step sizes of the discretized interface and the saturated growth speed further support our arguments.

There has been a growing interest in the interfaces which are self-affine fractals [1]. In these cases, the interface width $w(L,t)$ is expected to be a scaling function of the substrate size L and the growth time t in the form $w(L,t)\sim L^{\alpha}f(t/L^{z})$, such that $f(x)\to$ constant for $x\to\infty$ and $f(x)\sim x^{\alpha/z}$ for $x\to 0$. The roughening exponent α and the dynamical exponent z are of particular interest. Both computer simulation and theoretical analysis support the exact values $\alpha=\frac{1}{2}$ and $z=\frac{3}{2}$ in $1+1$ dimensions in a class of models.

However, recent experiments on immiscible fluid displacement gave $\alpha\simeq 0.73$ [2] and 0.81 [3] respectively and $\alpha/z\simeq 0.625$ [3], while a bacteria colony expansion experiment found $\alpha\simeq 0.78$ [4]. There have been several suggestions [5-8] to account for this. In particular, Zhang [5] proposed that this may be due to the existence of noise with a power law distribution instead of the usually assumed Gaussian form. Specifically, the proposed noise η has a distribution:

$$P(\eta)\sim\frac{1}{\eta^{1+\mu}}\qquad \eta>1. \tag{1}$$

Simulation in $1+1$ [5, 9] and $2+1$ dimensions [10] shows that the interface scales with exponents which are non-trivial functions of the parameter μ and the usual exponent identity [1]

$$\alpha+z=2 \tag{2}$$

holds [5, 9].

By assuming (2), Zhang [11] and Krug [12] independently derived

$$\alpha=\max\left\{\frac{d+2}{\mu+1},\frac{1}{2}\right\} \tag{3}$$

for $d+1$ dimensions. This predicts a critical value of $\mu=2d+3$ beyond which the power law noise is irrelevant and the usual values of the exponents for Gaussian noise case are restored. The physically interesting range of μ is $\mu\geqslant d+1$ so that $\alpha\leqslant 1$.

0305-4470/92/030135+06\$04.50

The values of α obtained from simulation [5, 9, 10] are consistently larger than that given by (3) while the agreement is better for μ close to $d+1$. In addition, the measured α approaches $\frac{1}{2}$ smoothly as μ is increased, favouring the non-existence of critical μ.

In this work we suggest an alternative microscopic description of interface growth with power-law noise in an attempt to give a deeper intuitive understanding of the growth process. For the case $\mu = d+1$ when $\alpha \simeq 1$ and the interface becomes a fractal, we shall demonstrate the existence of a scale invariant fractal pattern in the *bulk* of the aggregate and, in addition, prove the exponent identity (2). We note that although the identity has previously been proved through the continuum KPZ equation [13, 14], its validity becomes non-trivial as scale invariance implies that the interface is not smooth even at a macroscopic scale. Furthermore, it seems unclear theoretically whether the power-law noise can introduce an effective Gaussian component with power-law temporal correlation, in which case a violation of (2) is expected [14].

Let us examine the growth process more closely. We take a $(1+1)$-dimensional system for simplicity while the arguments can be generalized to higher dimensions. The specific model we adopt for simulation has been examined by Amar *et al* [9]. It is basically ballistic deposition defined by the discrete interface evolution equation,

$$h(i, t+1) = \max\{h(i, t) + \eta(i, t), h(i-1, t), h(i+1, t)\} \tag{4}$$

where $h(i, t)$ is the height of the aggregate at site i in the substrate co-ordinate at time t and every $\eta(i, t)$ is an independent random variable following the distribution $P(\eta)$ defined in (1). Periodic boundary conditions and a parallel updating algorithm are assumed so that h at all even (odd) sites will only evolve at even (odd) time steps. We can imagine that growth at a site i brought about by the addition of a particle of height $\eta(i, t)$ either stacked on top of the surface or stuck laterally to a neighbouring particle, leaving some empty space underneath.

We simulate the growth starting from a flat surface and examine the interface roughness only after the latter has fully developed. Figure 1 shows a typical aggregate at $\mu = 2$ in $1+1$ dimensions. A lot of empty spaces of various sizes are embedded inside the aggregate. By examining their distribution, we shall see in the following that the pattern in the bulk is indeed a fractal.

Consider the rare event when an $\eta(i, t)$ of exceptionally large value is sampled from (1) and creates a sharp peak at i on the interface. It is clear from (4) that growth by lateral sticking of particles is favoured at its edges. As a result, the peak widens rapidly with two cliff-like edges moving apart laterally at constant speed. Meanwhile, the rest of the interface behaves like a background on which the peak sits and advances, on average, with the saturated growth speed. The peak thus 'sinks' steadily into the background while expanding. Hence, the exceptional fluctuation η together with its lateral expansion gives an inverted triangle-like empty region which is finally immersed beneath the interface.

Many of these regions are indeed distorted due to 'collision' with others in the course when they grow. The pattern in the bulk is then an aggregate with empty spaces having a distribution of different shapes including those inverted triangles and their distortions. For simplicity, we will ignore the distorted ones and idealize them as a collection of inverted triangles of various sizes. Generalization to include a distribution of different shapes should not qualitatively affect our result.

Assume that the mean saturated growth speed of the interface is independent of the substrate size (we will further discuss this assumption in a moment). The slope of

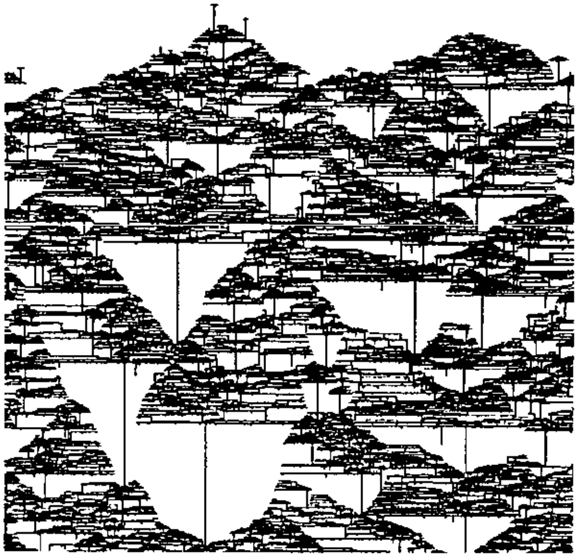

Figure 1. Snapshot of a growing aggregate for $\mu = 2$ and $L = 512$ in $1+1$ dimensions. Similar aggregates were also shown in [15].

the lower edges of the triangles then depends only on μ irrespective to the sizes of both the substrate and the triangle itself. Hence, they are similar triangles and can be rescaled to one another.

Let us now derive the scale invariance of the pattern. We pick out a square segment from the bulk of an aggregate of size $L \times L$, so that L defines a length scale. From the assumption of a fixed mean growth speed, the number of time steps it takes to grow the segment is $T \sim L$. The number of particles in the segment, and thus the number of random variables η involved is $N \sim LT \sim L^2$. We know that the height of a triangle simply equals the corresponding exceptional fluctuation $\eta(i, t)$ which creates the initial peak. The probability for a *single* random variable η to bring about a triangle of height in the range $[\xi L, (\xi + \mathrm{d}\xi)L]$ is proportional to $P(\xi L)(\mathrm{d}\xi)L$, where ξ and its infinitesimal change $\mathrm{d}\xi$ are constants independent of L. The total population $n(\xi)\,\mathrm{d}\xi$ of such triangles in the entire segment is given by

$$n(\xi)\,\mathrm{d}\xi \sim N \times P(\xi L)(\mathrm{d}\xi)L \sim \frac{\mathrm{d}\xi}{\xi^{\mu+1}L^{\mu-2}} \tag{5}$$

where (1) is used. Scale invariance is achieved if the pattern of triangles in different segments with various values of size L look alike in distribution after proper rescaling. This is in turn guaranteed if the population $n(\xi)\,\mathrm{d}\xi$ of triangles in the size range $[\xi L, (\xi + \mathrm{d}\xi)L]$, which scales with L, is independent of L. The condition is fulfilled at $\mu = 2$ and we thus have a scale invariant pattern in the bulk. In particular, the interface is a fractal and we have $\alpha = 1$. The time it takes to grow a triangle is proportional to its width, implying that time scales with length directly and hence $z = 1$. As a result, the exponent identity (2) follows for the $\mu = 2$ case.

We have explained the existence of sinking peaks which plays an important role in the growth process. We are going to demonstrate that they are also revealed in the power law tail of the probability distribution $\mathscr{P}(\delta)$ of the step sizes, δ, of the discretized interface, where we define δ at site i to be the height difference $h(i+2, t) - h(i, t)$

between next-neighbouring sites. (The height difference is not taken between nearest neighbours because they are not updated at the same time step.) Figure 2 shows the simulation result of $\mathscr{P}(\delta)$ against δ on a log-log scale. For more than a decade, $\mathscr{P}(\delta) \sim \delta^{-\theta}$, where $\theta \simeq |$ for $\mu \leqslant 5$.

Let us examine how the sinking of the peaks dictates the above power-law distribution. First note that practically every extraordinarily large step δ is due to the cliff-like edges of some laterally expanding peak. At the birth of a peak resulted from an exceptionally large noise $\eta > \delta$, it produces two steps of size η. As it sinks into the background with the mean satuated growth speed, the steps decrease at constant rate down to zero. Therefore it contributes to $\mathscr{P}(\delta)$ at the particular instance when the step size hits δ in the course of this decay process. Conversely, at any particular instance, every large step δ corresponds to a sinking peak resulted from a large noise $\eta > \delta$ occurred at a corresponding earlier time which depends on both δ and η. Integrating the contributions from all possible $\eta > \delta$, we have, using (1),

$$\mathscr{P}(\delta) \sim \int_{\delta}^{\infty} P(\eta)\, \mathrm{d}\eta \sim \frac{1}{\delta^{\mu}} \tag{6}$$

which is the desired result.

Finally, we come back to the assumption that the saturated growth speed, v, of the interface is independent of the lattice size L. Figure 3 plots the simulation result of v against L on semi-log scale. In $\mu = 2$ case,

$$v \sim \log L + \text{constant}. \tag{7}$$

That means v is a rather slowly increasing function of L. We expect that this speed of increase is slow enough to justify our assumption of constant growth speed as a first approximation. To demonstrate the self-consistency of our theory, we next apply

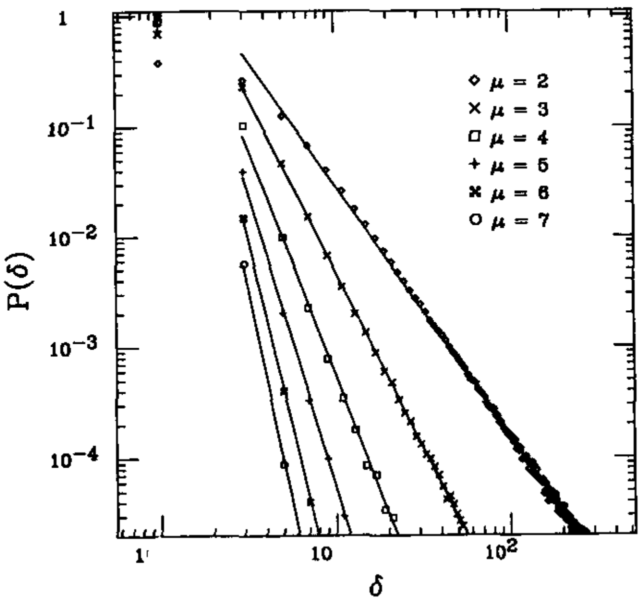

Figure 2. distribution $P(\delta)$ of step sizes, δ, against δ on semi-log scale. The fitted lines are in the form $P(\delta) \sim \delta^{-\theta}$, $\theta \simeq 0.28$, 3.18, 4.15, 5.37, 6.79 and 7.92 for $\mu = 2$–7.

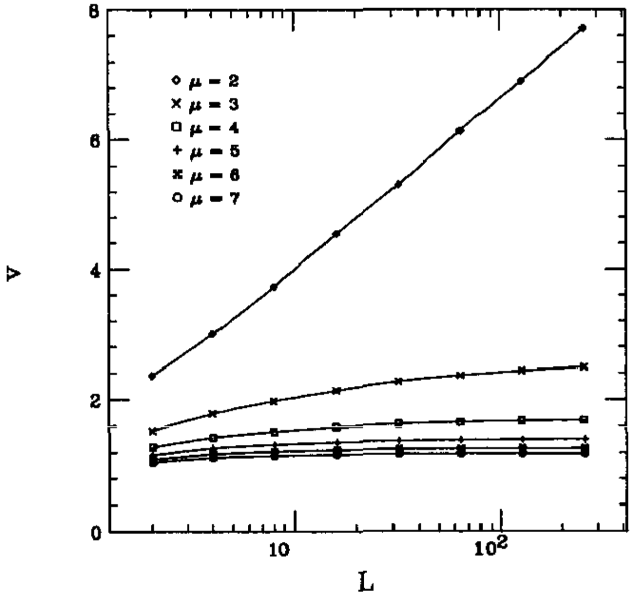

Figure 3. Saturated growth speed v (mean deposit height divided by number of layers of particles) against substrate size L.

our previous argument to deduce (7), which may be regarded as the first-order correction.

Consider a segment of deposit with a substrate size L grown for T time steps taken after saturation. There exists an upper cut-off Λ such that an inverted triangle formed by a large fluctuation $\eta > \Lambda$ would be wider than L and is too large to fit into the pattern. What will result is instead a truncated triangle, which does not contribute as much area to the segment as expected for a completed one. Now consider another similar segment grown on a substrate of size nL and for nT time steps, where $n > 1$ is an integer. The upper cut off then becomes $n\Lambda$. Let A and A' be the area of the two segments of sizes L and nL respectively. We now express A' in terms of A. First note that A' is the sum of two terms. One is the area of an assemble of n^2 smaller segments, which equals n^2A. The second term is the area of the extra triangles which can now be completed because of the increase of the cut off from Λ to $n\Lambda$. These triangles are formed by fluctuation $\eta \in [\Lambda, n\Lambda]$. Hence they have area and population proportional to η^2 and $(nL)(nT)P(\eta)$ respectively. Therefore,

$$A' = n^2 A + \int_{\Lambda}^{n\Lambda} \text{constant} \times \eta^2 \times (nL)(nT)P(\eta)\,\mathrm{d}\eta. \tag{8}$$

Applying (1) and integrating,

$$A' = n^2 A + \text{constant} \times n^2 LT \log n \tag{9}$$

for $\mu = 2$. We are interested in the saturated speed, v and v' for the two aggregates respectively. Because the heights of the segments are respectively A/L and $A'/(nL)$, we have

$$v = \frac{A}{LT} \qquad \text{and} \qquad v' = \frac{A'}{n^2 LT} \tag{10}$$

Substituting into (9) gives

$$v' = v + \text{constant} \times \log n \tag{11}$$

which is equivalent to (7).

Therefore it appears that the argument is self-consistent and the dependence of the saturated velocity on lattice size may well be treated as a perturbation. It may be interesting to investigate if this perturbation can account for the measured value of about 1.03 for the roughening exponent α in simulation [9] instead of the expected exact value 1.

We wish to thank D A Kessler and H Yan for many helpful discussions and D E Wolf for an enlightening conversation. This work is supported by NSF Grant DMR88-15908.

References

[1] Krug J and Spohn H 1990 *Solids Far From Equilibrium: Growth, Morphology and Defects* ed C Godrèche
Vicsek T 1989 *Fractal Growth Phenomena* (Singapore: World Scientific)
[2] Rubic M A, Edwards C A, Dougherty A and Gollub J P 1989 *Phys. Rev. Lett.* **63** 1685
Horváth V K, Family F and Vicsek T 1990 *Phys. Rev. Lett.* **65** 1388
Rubio M A, Dougherty A and Gollub J P 1990 *Phys. Rev. Lett.* **65** 1389
[3] Horváth V K, Family F and Vicsek T 1991 *J. Phys. A: Math. Gen.* **24** L25
[4] Vicsek T, Cserzó M and Horváth V K 1990 *Physica* A **167** 315
[5] Zhang Y-C 1990 *J. Physique* **51** 2129
[6] Amar J G, Lam P M and Family F 1991 *Phys. Rev.* A **43** 4548
[7] Kessler D A, Levine H and Tu Y 1991 *Phys. Rev.* A **43** 4551
[8] Martys N, Cieplak M and Robbins M O 1991 *Phys. Rev. Lett.* **66** 1058
[9] Amar J G and Family F 1991 *J. Phys. A: Math. Gen.* **24** L79
[10] Bourbonnais B, Kertész J and Wolf D E 1991 *J. Physique* II **1** 493
[11] Zhang Y-C 1990 *Physica* **170** 1
[12] Krug J 1991 *J. Physique* I **1** 9
[13] Meakin P, Ramanlal P, Sander L M and Ball R C 1986 *Phys. Rev.* A **34** 5091
[14] Medina E, Hwa T and Kardar M 1989 *Phys. Rev.* A **39** 3053
[15] Buldyrev S V, Havlin S, kertész J, Stanley H E and Vicsek T 1991 *Phys. Rev.* A **43** 7113